A note on the ensemble-averaged eigenvalue spectrum of large symmetric matrices

1977 J. Phys. A: Math. Gen. 10 L183

(http://iopscience.iop.org/0305-4470/10/11/002)

J. Phys. A: Math. Gen., Vol. 10, No. 11, 1977. Printed in Great Britain. © 1977

LETTER TO THE EDITOR

A note on the ensemble-averaged eigenvalue spectrum of large symmetric matrices

V K B Kota and V Potbhare

Physical Research Laboratory, Navrangpura, Ahmedabad 380 009, India

Received 11 August 1977

Abstract. We obtained an expression for ensemble-averaged eigenvalue spectrum for large symmetric matrices, where each matrix element is described by a Gaussian probability density function with same mean (non-zero) and variance. The eigenvalue density function is a sum of two semicircle distributions.

It is well known (Wigner 1955), that the Gaussian orthogonal ensemble of matrices gives rise to a semicircle distribution for eigenvalues. Each matrix element, in such ensembles is distributed according to a Gaussian probability density function with zero mean. Recently Edwards and Jones (1976) have considered ensembles with the same non-zero mean m for each matrix element. The eigenvalue density function they obtain depends very critically on the mean value of the matrix elements. It is identical to the semicircle distribution for $|m|<\lambda$, where λ is the critical value which depends upon the width of the semicircle distribution. For $|m|>\lambda$, the distribution they obtain is the sum of two distributions, one being a semicircle and the other a delta function outside the semicircle. They relate the sudden change of the eigenvalue density function when $|m|$ crosses the value λ to a phase transition. The derivation given by Edwards and Jones involves some approximations and tricks. The sudden change of the eigenvalue density function seems to be surprising from the point of view of the ensemble-averaged moments of the random matrices. In this letter using standard methods, we have obtained a general expression for ensemble-averaged moments of random symmetric matrices with the same non-zero mean for the matrix elements, and thereby the expression for the eigenvalue density function. It will be seen, looking ahead, that the eigenvalue density function changes smoothly as a function of m and satisfies the obvious elementary checks.

Consider a large $N\times N$ symmetric random matrix $\mathbf{R}$, each of whose matrix elements fluctuate about a fixed mean m with a probability density function

$$P(R_{ij})=\frac{1}{(2\pi\sigma^2)^{1/2}}\exp\left(-\frac{(R_{ij}-m)^2}{2\sigma^2}\right). \tag{1}$$

The matrix $\mathbf{R}$ can be written as $\mathbf{R}=m\mathbf{U}+\mathbf{R}_0$; where all the matrix elements of $\mathbf{U}$ are unity, and $\mathbf{R}_0$ is a random matrix with zero mean for its matrix elements. Therefore,

$$\mathbf{R}^p=(m\mathbf{U}+\mathbf{R}_0)^p=m^p\mathbf{U}^p+\sum_{q=1}^{p-1}\binom{p}{q}m^{p-q}\mathbf{U}^{p-q}\mathbf{R}_0^q+\mathbf{R}_0^p. \tag{2}$$

$\binom{p}{q}$ is a binomial coefficient equal to $p!/(p-q)!q!$. Since $\mathbf{U}^n = N^{n-1}\mathbf{U}$ for $n \geq 1$, we have

$$\mathbf{R}^p = m^p N^{p-1}\mathbf{U} + \sum_{q=1}^{p-1}\binom{p}{q} m^{p-q} N^{p-q-1}\mathbf{U}\mathbf{R}_0^q + \mathbf{R}_0^p. \tag{3}$$

Therefore

$$\langle \mathbf{R}^p \rangle = (\mathrm{Tr}\mathbf{R}^p)/N = m^p N^{p-1} + \frac{\mathrm{Tr}(\mathbf{R}_0^p)}{N} + \sum_{q=1}^{p-1}\binom{p}{q} m^{p-q} N^{p-q-1}\frac{\mathrm{Tr}(\mathbf{U}\mathbf{R}_0^q)}{N}. \tag{4}$$

We know that

$$\mathrm{Tr}(\mathbf{U}\mathbf{R}_0^q) = \sum_{i,j=1}^{N} (\mathbf{R}_0^q)_{ij}. \tag{5}$$

Since we are interested in ensemble-averaged moments (indicated by a bar)

$$\overline{(\mathbf{R}_0^q)_{ij}} = \overline{(\mathbf{R}_0^q)_{ij}}\delta_{ij} \qquad \text{for all } q. \tag{6}$$

This follows very simply from the arguments of Mon and French (1975). Hence,

$$\overline{\langle \mathbf{R}^p \rangle} \equiv M_p = m^p N^{p-1} + M_p^0 + \sum_{q=1}^{p-1}\binom{p}{q} m^{p-q} N^{p-q-1} M_q^0 \tag{7}$$

where M_q^0 is the ensemble-averaged qth moment of R_0, which corresponds to the qth moment of the semicircle distribution of Wigner (1955). By slight re-arrangement of the terms in equation (7), we obtain

$$M_p = \frac{N-1}{N} M_p^0 + \frac{1}{N}\sum_{q=0}^{p}\binom{p}{q} m^{p-q} N^{p-q} M_q^0. \tag{8}$$

The eigenvalue density function which is generated by the above moments (M_p) can be written down trivially as

$$\rho(x) = \frac{N-1}{N}\rho_0(x) + \frac{1}{N}\rho_0(x - mN) \tag{9}$$

where $\rho_0(x)$ is a semicircle distribution giving rise to moments $M_q^0 = \int \rho_0(x) x^q \, dx$ with $M_0^0 = 1$. This is a smoothly varying function of m and for $m = 0$, $\rho(x) = \rho_0(x)$. For any value of m, the mean of $\rho(x) = m$ which is equal to $\overline{\langle \mathbf{R} \rangle}$. Also for $\sigma \to 0$, each matrix element of $\mathbf{R}$ tends to the value m, then

$$\rho(x) \to \frac{N-1}{N}\delta(x) + \frac{1}{N}\delta(x - mN);$$

which is a standard result.

Thus, the result given in equation (9) indicates that there is no sudden change of eigenvalue density function with respect to m. It also shows that the state that moves out is not located at a particular energy under ensemble averaging but the probability of it being found at a certain energy is given by a semicircle distribution, the width of which is the same as the $\rho_0(x)$. We anticipate a similar result (as in equation (9)) for the two-body random ensembles of French and Wong (1970), where the semicircle would be replaced by a Gaussian for a large number of particles. The results of this work will be reported in the near future.

We thank Professor J C Parikh for bringing the article by Edwards and Jones (1976) to our attention. V K B Kota would like to thank CSIR (India) for financial support.

References

Edwards S F and Jones R C 1976 *J. Phys. A: Math. Gen.* **9** 1595
French J B and Wong S S M 1970 *Phys. Lett.* **33B** 449
Mon K K and French J B 1975 *Ann. Phys., NY* **95** 90
Wigner E P 1955 *Ann. Math., Lpz.* **62** 548